How fast travelling waves can attract small initial data

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Model under study

$$\partial_t u - D\partial_{xx}^2 u = v - \mu u$$

$$d\partial_y v = \mu u - v$$

$$\partial_t v - d\Delta v = f(v)$$

$$-\partial_y v = 0 \tag{1}$$

where $f(v)$ is of the ignition type.

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- Motivation : robustness of the propagation enhancement discovered by BRR.

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$$\begin{array}{lcr}
0 \leftarrow u & -u'' + c_\infty u' = v - \mu u & u \to 1/\mu \\
\hline
 & d\partial_y v = \mu u - v & \\
0 \leftarrow v & c_\infty \partial_x v - d\partial_{yy}^2 v = f(v) & v \to 1 \\
 & \partial_y v = 0 & \\
\hline
\end{array} \tag{2}$$

which is well posed.

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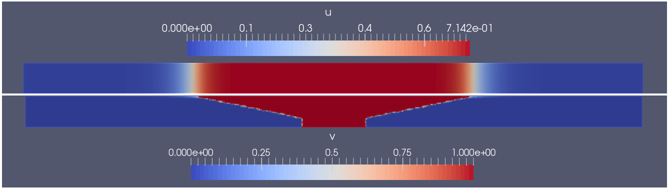

Enhancement by diffusion : the homogeneous case

$$\begin{cases} \partial_t v - \partial^2_{xx} v = f(v) & t > 0, x \in \mathbb{R} \\ v_0(x) = \mathbf{1}_{(-L,L)}(x) \end{cases} \tag{3}$$

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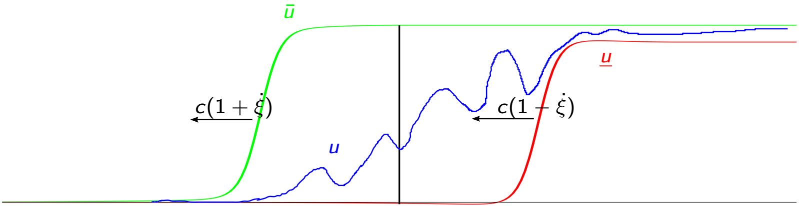
Back to our system : what happens ?

Theorem 1

Let (u_0, v_0) be front-like. There exists $\omega > 0$ indep. of D s.t. for $\varepsilon > 0$ small there exists two shifts $\xi_1^{\pm}$ s.t.

$$\begin{aligned} \phi(x + c\xi_1^- + ct) - C\varepsilon e^{-\omega t} &\leq \mu u(t,x) \ \leq \mu\phi(x + c\xi_1^+ + ct) + C\varepsilon e^{-\omega t} \\ \psi(x + c\xi_1^- + ct) - C\varepsilon e^{-\omega t} &\leq v(t,x,y) \leq \psi(x + c\xi_1^+ + ct) + C\varepsilon e^{-\omega t} \end{aligned}$$

where $C = C(d)$.



Consequence

Theorem 2

Let (u_0, v_0) *be* ≥ 0 *smooth and compactly supported. There exists* $\delta > 0$ *and* $M > 0$ *indep. of* D *such that if*

$$\mu u_0, v_0 > 1 - \delta \text{ for } x \in (-M\sqrt{D}, M\sqrt{D})$$

then $\mu u, v$ *stays trapped (up to an exponentially decaying error) between two shifts of a pair of travelling waves evolving in both directions.*

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Idea of proof.

- Upper bound : $(\min(\bar{u}, \tilde{u}), \min(\bar{v}, \tilde{v}))$ is a supersolution ($\tilde{u}$ is like $\overline{u}$ with reversed x). Can be put above (u_0, v_0) at inital time.

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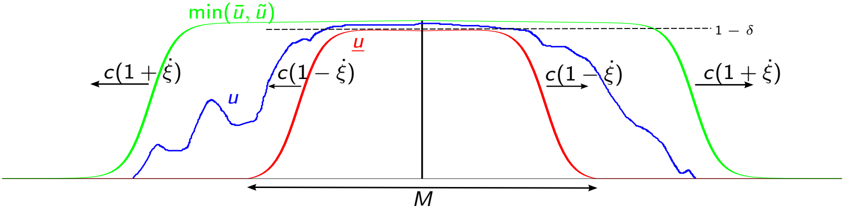
- Lower bound (idea of FML) :

$$\begin{cases} \underline{u} = \max\left(0, \phi + \tilde{\phi} - 1/\mu - q_u(t)/\mu \min(\Gamma, \tilde{\Gamma})\right) \\ \underline{v} = \max\left(0, \psi + \tilde{\psi} - 1 - q_v(t, y) \min(\Gamma, \tilde{\Gamma})\right) \end{cases}$$

□

Subsolution provided initial shifts are large enough.



M and δ arise when one wants to put $(\underline{u}, \underline{v})(0)$ below (u_0, v_0).

What about small initial data when D is large ?

Theorem 3

There exists $M', \delta' > 0$ independent of $D > d$ such that if

$$v_0 > 1 - \delta' \text{ for } x \in (-M', M')$$

then after a time $t_D = D^{1/2} \ln D + O(1)$ one has μu and v satisfying the assumptions of Theorem 2.

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- (4) has a steady state $p(y) > 1 - \delta$ (provided L is not too small). Berestycki–Nirenberg '92 : $\exists$ T.W. connecting 0 and $p(y)$ with speed c_p indep. of D. This gives :

$\square$

Lemma 4 (Behaviour at the bottom)

Under the assumptions of Theorem 3, there holds

$$v(t,x,-L) \geq (1-\delta'')\varphi_t(x) - Ce^{-bt}$$

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Rescale by $x \leftarrow x/\sqrt{D}$. Goal :

$$\liminf_{t\to+\infty} \inf_{D>d} \min_{(x,y)\in\overline{\Omega_{L,M}}} \{\mu u^D(T_D+t,x), v^D(T_D+t,x,y)\} \geq p(-L) > 1-\delta \quad (5)$$

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- Easy but tedious : LHS of (5) can be characterised as lim of $\mu u^{D_n}(T_{D_n}+t_n, x_n)$ or $v^{D_n}(T_{D_n}+t_n, x_n, y_n)$ where $t_n \to +\infty$, $D_n > d$, $(x_n, y_n) \in \overline{\Omega_{L,M}}$. Extract so that (D_n) has a limit in $[d, +\infty]$.

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Idea : $\underline{v}_n(t, x, -L) \underset{n \to +\infty}{\to} p(-L) > 1 - \delta$ loc. unif. in $\mathbb{R} \times \mathbb{R}$ (either (D_n) bounded and it is immediate, or since $\ln(D_n) \to +\infty$)

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- Regularity in y is OK by rescaling.
- Regularity in x falls but : (6) is linear and (φ_t) bounded in $\mathcal{C}^3$, so use the maximum principle.

Now extract : (u_∞, v_∞) **global in time** (since $t_n \to +\infty$) :

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The maximum principle applies in the standard way for u and on every y-slice for v : $\mu u_\infty, v_\infty \equiv p(-L)$ and this proves (5) and thus Theorem 3.

Additional information

Initial datum supported on the road : $v_0 \equiv 0, \mu u_0 = \mathbf{1}_{(-L,L)}(x)$

Theorem 5

There exists a_0, a_1 and $\mu^{\pm}$ indep. of D such that

- *If $L < a_0\sqrt{D}$, extinction occurs.*
- *If $L > a_1\sqrt{D}$, invasion occurs if $\mu < \mu-$ and extinction if $\mu > \mu^+$.*

Merci pour votre attention !