21-268 – Homework assignment week #4

Laurent Dietrich
Carnegie Mellon University, Spring 2017, Sec. A and B

Reminder

Homework will be given on Wednesdays and due on the next Wednesday before 5pm, to me in class or in Christopher Cox's mailbox in Wean Hall 6113 (pay attention to the arrow !). Late homework will never be accepted without a proper reason. In case of physical absence, electronic submissions by e-mail to both me your TA can be accepted. Please do not forget to write your name, andrew id and section and please use a staple if you have several sheets.

Reading

1. Proofs in the textbook that I did not do in class.

Exercises (22 pts)

1. (1+3 pts) Let
$$\vec{f}(x,y,z) = \begin{pmatrix} xy^2z^3e^{\sqrt{xy}} \\ xy\sin(yz) \end{pmatrix}$$
On what domain does $\vec{f}$ have continuous partial derivatives ? Compute the Jacobian matrix of $\vec{f}$ there.

2. Let
$$F(x,y) = \begin{pmatrix} e^x\cos(y) \\ e^x\sin(y) \end{pmatrix}$$
   (a) (2 pts) Prove that around every $(x,y) \in \mathbb{E}^2$ there exists an open neighborhood on which F is invertible.
   (b) (2 pts) Is F invertible on the whole $\mathbb{E}^2$?

3. Let $f(x,y,z) = x^2 + y^2 + z^2 - C(xy + xz + yz)$
   (a) (1 pt) Show that $\nabla f(0,0,0) = 0$. *We say that* $(0,0,0)$ *is a critical point for* f*.*
   (b) (2 pt) Find a value of C for which this critical point is also a relative minimum point.
   (c) (2 pts) Find a value of C for which it is not a relative minimum point.

4. Let $f(x,y) = e^x\sin(y)$.
   (a) (2 pts) Call $\vec{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. What is the slope of the graph of f along the direction $\vec{n}$ at (x,y) ?
   (b) (2 pts) Show that $\Delta f = 0$. *We say that* f *is a harmonic function.*

5. (5 pts) Let $f : \mathbb{E}^n \to \mathbb{E}^1$ with continuous partial derivatives up to order 2 satisfy
$$f(x) = g(|x|)$$
for some $g : \mathbb{E}^1 \to \mathbb{E}^1$ twice differentiable. That is, the value of f depends only on the radial variable $|x|$ (in other words the graph of f has radial symmetry along the z axis). Prove that
$$\Delta f(x) = g''(r) + \frac{n-1}{r}g'(r) \quad \text{where } r = |x| = \sqrt{x_1^2 + \cdots + x_n^2}$$
for all $x \in \mathbb{E}^n \setminus \{0\}$. *Hint: start by computing* $\frac{\partial^2 f}{\partial x_1^2}$.