21-268 – Homework assignment week #13

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Reminder

Homework is due next Wednesdays before 5pm, to me in class or in Christopher Cox's mailbox in Wean Hall 6113 (pay attention to the arrow !). Late homework will never be accepted without a proper reason. In case of physical absence, electronic submissions by e-mail to both me your TA can be accepted. Please do not forget to write your name, andrew id and section and please use a staple if you have several sheets.

Exercises (24 pts)

1. Determine if $\int P\mathrm{d}x + Q\mathrm{d}y$ is path independent in the following situations:
   (a) (2 pts) $P = \frac{x^3}{x^4+y^2}, Q = \frac{1}{2}\frac{y}{x^4+y^2}$ and $D = \mathbb{E}^2 \setminus \{(0,0)\}$.
   (b) (2 pts) $P = \frac{x+y}{x^2+y^2}, Q = \frac{y-x}{x^2+y^2}$ and $D = \mathbb{E}^2 \setminus \{(x,0) \mid x \leq 0\}$.
   (c) (2 pts) $P = \frac{x+y}{x^2+y^2}, Q = \frac{y-x}{x^2+y^2}$ and $D = \mathbb{E}^2 \setminus \{(0,0)\}$.

2. (6 pts) Let $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$ and
$$S = \{(x,y,z) \mid x^2+y^2 = 4,\ 0 \leq z \leq 1\}$$
oriented towards the z-axis. Compute
$$\iint_S \vec{F} \cdot \mathrm{d}\vec{\sigma}.$$

3. (6 pts) This problem is an illustration of the divergence theorem. Let $R_0 > 0$,
$$R = \{(x,y,z) : x^2+y^2+z^2 \leq R_0^2\} \quad \text{and} \quad S = \{(x,y,z) : x^2+y^2+z^2 = R_0^2\}$$
with outward normal. Let $\vec{F}(x,y,z) = x\vec{i} + y\vec{j}$. Compute
$$\iint_S \vec{F} \cdot \mathrm{d}\vec{\sigma} \quad \text{and} \quad \iiint_R \vec{\nabla} \cdot \vec{F} \mathrm{d}V$$
separately and check that they are equal.

4. (6 pts) Let R be a volume bounded by a surface S to which the divergence theorem applies. Let $f(x,y,z)$ define a scalar valued $\mathcal{C}^1$ function. Use the divergence theorem to show that
$$\iiint_R \vec{\nabla} f \mathrm{d}V = \iint_S f\vec{n}\mathrm{d}\sigma$$
where $\vec{n}$ is the outward unit normal.