21-241 – Handout on an introduction to eigenvalues

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1 The Fibonacci sequence

Let $F_0 = 0,\ F_1 = 1$ and define a recurring sequence by

$$\forall n \in \mathbb{N}^*,\ F_{n+1} = F_n + F_{n-1}.$$

This is the celebrated *Fibonacci* sequence $0, 1, 1, 2, 3, 5, 8, 13, ...$ Can we compute the value of F_n for all $n \in \mathbb{N}$ without doing the whole summation ? Observe that this recurrence relation can be written as follows:

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

More precisely it is encoded in the first row of this equality, and the second one just states $F_n = F_n$. The advantage of this formulation is the following

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \cdots = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So that if one can compute A^n, one can compute F_n. This is where *eigenvalues* and *eigenvectors* appear.

Call

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \qquad S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The λ are *eigenvalues* of A and the columns of S are associated *eigenvectors*. By the end of the chapter, you will be able to find these by yourselves. Now observe that

$$A = SDS^{-1}$$

So $A^2 = SDS^{-1}SDS^{-1} = SD^2S^{-1}$. Similarly,

$$A^n = SD^nS^{-1}$$

And the n^{th} power of a diagonal matrix is easy to compute. Exercise: check that this allows you to compute

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

2 Other applications

Eigenvalues and eigenvectors are very natural objects. An eigenvector of a linear transformation (and by extension of a matrix) is a vector whose direction does not change when transformed. The associated eigenvalue is the length multiplier. For instance, the axis of a rotation is an eigenvector for it and the eigenvalue is 1. Due to their nature and to their deep mathematical meaning, they appear everywhere. Here is a non-exhaustive list of examples where they are found important.

- Mechanics: finding the principal axes of a body and associated moments of inertia is exactly finding eigenvectors and eigenvalues of a matrix.
- Physics, chemistry: coupled systems of linear differential equations (for instance to describe concentrations of several chemicals reacting) may be solved by *diagonalization* (i.e. exactly what we did above).
- Data analysis: Principal component analysis is an application of diagonalization. Google's Pagerank algorithm looks for specific eigenvalues of very large matrices.
- Economics: Leontief's model of a government's economy looks for equilibria by investigating eigenvalues.