Homework #9 19/19 4 November 2016

EXERCISE 1

1. $\begin{bmatrix} 1/8 & 1/8 & 1/8 \\ -1/4 & 3/4 & -1/4 \\ 5/8 & -3/8 & -3/8 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & 3/4 & 3/4 \\ 1/2 & -1/2 & 1/2 \\ -3/4 & 3/4 & 3/4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ✓

2. Since the matrix $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix}$ above is diagonizable, good!

i.e. the inverse of $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{} \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow[3R_2 - 2R_1]{} \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 3 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow[3R_3 + R_2]{} \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 3 & 0 \\ 0 & 0 & -8 & -5 & 3 & 3 \end{array}\right] \xrightarrow[4R_2 - R_3]{} \left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 12 & 0 & -3 & 9 & -3 \\ 0 & 0 & -8 & -5 & 3 & 3 \end{array}\right] \xrightarrow[8R_1 + R_3]{} \left[\begin{array}{ccc|ccc} 24 & 0 & 0 & 3 & 3 & 3 \\ 0 & 4 & 0 & -1 & 3 & -1 \\ 0 & 0 & -8 & -5 & 3 & 3 \end{array}\right]$

$\xrightarrow[\frac{1}{24}R_1,\ \frac{1}{4}R_2,\ -\frac{1}{8}R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/8 & 1/8 & 1/8 \\ 0 & 1 & 0 & -1/4 & 3/4 & -1/4 \\ 0 & 0 & 1 & 5/8 & -3/8 & -3/8 \end{array}\right]$ ✓ $= \begin{bmatrix} 1/8 & 1/8 & 1/8 \\ -1/4 & 3/4 & -1/4 \\ 5/8 & -3/4 & -3/8 \end{bmatrix}$ so $P^{-1}AP = D$, Therefore

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the columns of P represent eigenvectors of A, and the diagonal entries of D the corresponding eigenvalues.

Thus, the eigenvalues of $\begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix}$ are 6 and -2, with

$E_6 = \text{span}\left\{\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}\right\}$ and $E_{-2} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$. ✓

EXERCISE 2

1. First we find the eigenvalues of A.

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda + \lambda^2 - 1) - (1-\lambda)$

$= (1-\lambda)(\lambda^2 - \lambda - 2)$

$= (1-\lambda)(\lambda - 2)(\lambda + 1)$

Therefore, the eigenvalues are 1, -1, and 2. ✓

we see that:

$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \longleftrightarrow R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 + x_2 = x_3$, $x_1 = -x_2$, $x_3 = 0$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ ✓

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$E_{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[2R_3 - R_1]{} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $2x_1 = -x_3$, $2x_2 = -x_3$ $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$ ✓

$E_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_3 + R_1]{} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 + R_2]{} \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $x_1 = x_3$, $x_2 = x_3$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ✓

So $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

2. The above process works because A has n linearly independent eigenvectors, so it is diagonizable. ✓

EXERCISE 3

1. (a) $A \sim A$

We see that $I_n^{-1} A I_n = A$, so by definition $A \sim A$.

(b) $A \sim B \Rightarrow B \sim A$.

Assume $A \sim B$. That is, there exists an invertible matrix P s.t.

$$P^{-1}AP = B.$$

We can then multiply by P to obtain

$$PP^{-1}AP = PB$$ and then by P^{-1} to obtain

$$PP^{-1}APP^{-1} = PBP^{-1}.$$

$PP^{-1} = I_n$, so we have $I_n A I_n = PBP^{-1}$, so $A = PBP^{-1}$

By definition, $B \sim A$.

(c). $A \sim B \wedge B \sim C \Rightarrow A \sim C$

Let $P^{-1}AP = B$ and $Q^{-1}BQ = C$. Substituting the first equation into the second, we arrive at

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$$Q^{-1}P^{-1}APQ = C$$

Properties of inverse matrices tell us that the above is equivalent to

$$(PQ)^{-1} A (PQ) = C,$$ so by definition $A \sim C$.

2. Suppose a matrix A is similar to I_n. Then, there exists an invertible matrix P such that $AP = PI_n$. But then $AP = P$, so A must be I_n. Therefore, the only matrix similar to I_n is I_n itself.

3. If 2 matrices are similar, their determinants are equal. But we see that $\det(A) = 1$ and $\det(B) = -5$ so A and B are not similar.

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