Matrices HW #7.

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Ex 1

(a) $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$\xrightarrow{R_3 \leftarrow R_3/(-2)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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Basis for row$(A) = \{[1\ 0\ 1\ 0],\ [0\ 1\ -1\ 0],\ [0\ 0\ 0\ 1]\}$

Basis for col$(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ let $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$. $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Thus let $x_3 = t$, $x = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix}$ Thus, the basis for null$(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) $A = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_3} \begin{bmatrix} 1 & -2 & -1 & -2 & -3 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_3}$

$\begin{bmatrix} 1 & -2 & -1 & -2 & -3 \\ 0 & 0 & 2 & 6 & 7 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & -2 & -1 & -2 & -3 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 2 & 6 & 7 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2}$

$\begin{bmatrix} 1 & -2 & -1 & -2 & -3 \\ 0 & 0 & 2 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2/2} \begin{bmatrix} 1 & -2 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 + R_2}$

$\begin{bmatrix} 1 & -2 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 3 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Basis for row$(A) = \{[1\ -2\ 0\ 1\ \tfrac{1}{2}],\ [0\ 0\ 1\ 3\ \tfrac{7}{2}]\}$

Basis for col$(A) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$[A|0] = \left[\begin{array}{ccccc|c} 2 & -4 & 0 & 2 & 1 & 0 \\ -1 & 2 & 1 & 2 & 3 & 0 \\ 1 & -2 & 1 & 4 & 4 & 0 \end{array}\right] \sim\!\!\to \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$, let $x_2 = d$, $x_4 = e$, $x_3 = f$ then

$x_1 = 2d - e - \frac{1}{2}f$; $x_3 = -3e - \frac{7}{2}f$.

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Thus $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2d - e - \frac{1}{2}f \\ d \\ -3e - \frac{7}{2}f \\ e \\ f \end{bmatrix} = d\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e\begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + f\begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix}$

Basis of $null(A) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$.

Ex 2

Basis of $span\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right)$ = Basis of $col\left(\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \right)$

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$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 3 \\ 1 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow -R_3} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_3} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow (R_3 - R_2)\cdot(-1)} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

the basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Ex 3

1. No, there should be three vectors in the basis since $\dim(\mathbb{R}^3) = 3$.
2. Since $\dim(row(A)) \leq 3$, and $\dim(col(A)) = \dim(row(A))$, $\dim(col(A)) \leq 3$, according to the definition of basis and dimension, at most 3 of the columns of A are linearly independant. Thus the 5 columns of A must be linearly dependant.
3. Similarly to #2, $\dim(row(A)) = \dim(col(A)) \leq 2$, then at most 2 of the 4 rows are linearly independant, which means the 4 rows of A must be linearly dependent.

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4. Since rank (A) + nullity (A) = n, and if A is not a zero matrix, then $1 \le$ rank (A) ≤ 3, n = 5 ; thus $2 \le$ nullity (A) ≤ 4.
If A is a zero matrix, then nullity (A) = n = 5.
Thus possible values of nullity (A) is {2, 3, 4, 5}

5. Since rank (A) + nullity (A) = n, n = 2, and $0 \le$ rank (A) ≤ 2
$0 \le$ nullity (A) ≤ 2, thus possible values are {0, 1, 2}

Ex 4

Let matrix $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + R_2}$

$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 + R_3} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 \times (-1)} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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According to the fundamental theorem, rank (A) = 3, while n = 4, thus the column vectors of A are not linearly independent,
Therefore, the four vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\boxed{\text{don't}}$ form a basis of $\mathbb{R}^4$ according to the definition of basis.