- Homework 5

14/14

Let $A \in M_{mn}(\mathbb{R}), B \in M_{nm}(\mathbb{R})$
Let $i, j \in \mathbb{N}$ st $1 \leq i \leq m \quad 1 \leq j \leq n$

for i – iii

i) NTS $((A^T)^T)_{ij} = (A)_{ij}$

$((A^T)^T)_{ij}$
$= (A^T)_{ji}$ by the definition of transpose
$= A_{ij}$ by the definition of transpose

Also for $\lambda \in \mathbb{K}$

ii) NTS $((\lambda A)^T)_{ji} = \lambda (A^T)_{ji}$

$((\lambda A)^T)_{ji}$
$= (\lambda A)_{ij}$ by the definition of transpose
$= \lambda (A_{ij})$ by associativity of scalar multiplication
$= \lambda (A^T)_{ji}$ by the definition of transpose

iii) NTS $((A+B)^T)_{ji} = (A^T)_{ji} + (B^T)_{ji}$

$((A+B)^T)_{ji}$
$= (A+B)_{ij}$ by the definition of transpose
$= A_{ij} + B_{ij}$
$= (A^T)_{ji} + (B^T)_{ji}$ by the definition of transpose

1 3 6 10
1 2 3 4

2) a) $B^2 = BB = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

$B^3 = B^2 B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✓

b) $A = B + I_3$

because $\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{B} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{A}$ ✓

c) $A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = A^2 A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$A^4 = A^3 A = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$ for all $n >= 1$

✓ PROOF STARTS HERE, By Induction

Base case: $n = 1$ $A^1 = \begin{bmatrix} 1 & (1) & \frac{(1)(1+1)}{2} \\ 0 & 1 & (1) \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = A$ ✓

IH: assume $A^k = \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ for some $k >= 1$

NTS the formula holds for some $n = k+1$

By the definition of matrix powers and the IH,

$$A^{k+1} = A^k A = \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+k & 1+k+\frac{k(k+1)}{2} \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (k+1) & \frac{(k+1)2}{2} + \frac{k(k+1)}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (k+1) & \frac{2k+2 \; +k^2+k}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (k+1) & \frac{(k+1)(k+2)}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (k+1) & \frac{(k+1)((k+1)+1)}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the formula holds for all $n >= 1$ by induction