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21-241 Homework Assignment Week #2

1. The proof states that rows 1 and 2 are identical after performing $R_2 \leftarrow R_2 + R_1$ and $R_1 \leftarrow R_1 + R_2$, but that is false. R_2 changes in the first step, so in the second step, we add a different "version" of R_2. Therefore, we cannot say that R_1 and R_2 are identical.

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2. $$\begin{cases} -x_1 + 3x_2 - 2x_3 + 4x_4 = 0 \\ 2x_1 - 6x_2 + x_3 - 2x_4 = -3 \\ x_1 - 3x_2 + 4x_3 - 8x_4 = 2 \end{cases}$$

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \xrightarrow{2R_1+R_2} \left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right]$$

$$\xrightarrow{R_1+R_3} \left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{cccc|c} -1 & 3 & 0 & 0 & 2 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2,\ \frac{1}{2}R_3} \left[\begin{array}{cccc|c} -1 & 3 & 0 & 0 & 2 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{cccc|c} -1 & 3 & 0 & 0 & 2 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} -x_1 + 3x_2 = 2 \\ -x_3 + 2x_4 = -1 \end{cases} \qquad \begin{aligned} x_2 &= s \\ x_4 &= t \end{aligned}$$

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$$x_3 = 2x_4 + 1 = 2t + 1$$
$$x_1 = 3x_2 - 2 = 3s - 2$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \left\{ \begin{pmatrix} 3s-2 \\ s \\ 2t+1 \\ t \end{pmatrix} \middle/ t, s \in \mathbb{R} \right\}}$$

✓

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Pg. 12 Ex. 1, 3 & Pg. 21 EX 2

3. (a) For every integer x, if x is even, then for every integer y, xy is even.

$$\forall x \in \mathbb{Z}, (x \text{ is even}) \Rightarrow (\forall y \in \mathbb{Z}, xy \text{ is even})$$

Proof. Let $x, y \in \mathbb{Z}$.

Assume x is even. By definition of even integers, x can be expressed as

$x = 2k$ for some $k \in \mathbb{Z}$

$\Rightarrow xy = 2ky$ by substituting $2k$ for x

Because ky is an integer by properties of integer multiplication, xy is even for all y when x is even by definition of even integers. $\square$

(b) For every integer x and for every integer y, if x is odd and y is odd then $x+y$ is even.

$$\forall x, y \in \mathbb{Z}, [(x \text{ is odd}) \wedge (y \text{ is odd})] \Rightarrow (x+y \text{ is even})$$

Proof. Let $x, y \in \mathbb{Z}$

Assume x is odd and y is odd. By definition of odd integers, x and y can be represented as

$x = 2m+1$ for some $m \in \mathbb{Z}$

and $y = 2n+1$ for some $n \in \mathbb{Z}$

Thus,

$x+y = (2m+1)+(2n+1)$ by substitution

$= 2m+2n+2$

$= 2(m+n+1)$ by factoring out 2

Because $(m+n+1)$ is an integer by integer arithmetic, $(x+y)$ is even when x and y are both odd by definition of even integers. $\square$

(c) For every integer x, if x is odd then x^3 is odd.

$$\forall x \in \mathbb{Z}, (x \text{ is odd}) \Rightarrow (x^3 \text{ is odd})$$

Proof. Let $x \in \mathbb{Z}$.
Assume x is odd. By definition of odd integers, x can be expressed as

$$x = 2k+1 \text{ for some } k \in \mathbb{Z}$$

Thus,

$$x^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 \quad \text{by distribution}$$
$$= 2(4k^3 + 6k^2 + 3k) + 1 \quad \text{by factoring out } 2$$

Because $(4k^3+6k^2+3k)$ is an integer by integer arithmetic properties, x^3 is odd when x is odd by definition of odd integers. □

The negation of each of the statements:

(a) $\neg[\forall x \in \mathbb{Z}, (x \text{ is even}) \Rightarrow (\forall y \in \mathbb{Z}, xy \text{ is even})]$

$\boxed{\exists x \in \mathbb{Z}, (x \text{ is even}) \wedge (\exists y \in \mathbb{Z}, xy \text{ is odd})}$

(b) $\neg[\forall x, y \in \mathbb{Z}, [(x \text{ is odd}) \wedge (y \text{ is odd})] \Rightarrow (x+y \text{ is even})]$

$\boxed{\exists x, y \in \mathbb{Z}, [(x \text{ is odd}) \wedge (y \text{ is odd})] \wedge (x+y \text{ is odd})}$

(c) $\neg[\forall x \in \mathbb{Z}, (x \text{ is odd}) \Rightarrow (x^3 \text{ is odd})]$

$\boxed{\exists x \in \mathbb{Z}, (x \text{ is odd}) \wedge (x^3 \text{ is even})}$

4. $(\forall x \in \mathbb{Z})((\exists y \in \mathbb{Z})\, x = 3y+1) \Rightarrow ((\exists y \in \mathbb{Z})\, x^2 = 3y+1)$

Proof. Let $x, \cancel{y} \in \mathbb{Z}$. Assume $\exists\, y \in \mathbb{Z}$ such that –

~~Assume~~ $x = 3y+1$.

Then, $x^2 = (3y+1)^2$

$$= 9y^2 + 6y + 1 \quad \text{by distribution.}$$

This can be rewritten as

$$x^2 = 3(3y^2 + 2y) + 1 \quad \text{by factoring out a } 3.$$

Because this is of the form $3y+1$ and $(3y^2+2y)$ is an integer, the logical formula is true. □

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5. Guess a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

& prove by induction.

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$n=1$ $\quad \frac{1}{2}$

$n=2$ $\quad \frac{1}{2}+\frac{1}{6}=\frac{2}{3}$

$n=3$ $\quad \frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}$

Hypothesis: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{n(n+1)}=\frac{n}{n+1}$

Proof. Base Case: $n=1$

$$\frac{1}{1\cdot 2}+\cdots+\frac{1}{n(n+1)}=\frac{1}{2}$$

and $\frac{n}{n+1}=\frac{1}{2}$

Inductive Step. Suppose that for a given $n\in\mathbb{N}$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1} \quad \text{(inductive hypothesis)}$$

We need to show that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}$$

If we add $\frac{1}{(n+1)(n+2)}$ to both sides of the inductive hypothesis, we get

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$$

Simplifying the right side, we get

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)+1}{(n+1)(n+2)}$$

$$=\frac{n^2+2n+1}{(n+1)(n+2)}$$

$$=\frac{\cancel{(n+1)}(n+1)}{\cancel{(n+1)}(n+2)}$$

$$=\frac{n+1}{n+2}$$

Therefore, $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots\frac{1}{n(n+1)}=\frac{n}{n+1}$. $\square$