1) • $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $[x]_B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. $C = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$

One could see that $x = 1 \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (or if not just solve $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{array}\right]$) so

$$\boxed{[x]_C = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}}$$

On the other hand

$P_{B\leftarrow C} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. By using Gauss-Jordan method one can compute

$$P_{C\leftarrow B} = \left(P_{B\leftarrow C}\right)^{-1} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}$$

So we have $[x]_C = P_{C\leftarrow B}[x]_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ indeed.

• $x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ $B = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

$x = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $[x]_B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

$C = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$. $\left[\begin{array}{cc|c} 0 & 2 & 4 \\ 1 & 3 & -1 \end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 2 & 4 \end{array}\right]$

$\longleftrightarrow \left[\begin{array}{cc|c} 1 & 3 & -1 \\ 0 & 1 & 2 \end{array}\right]$

$\longleftrightarrow \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & 2 \end{array}\right]$

so $[x]_C = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$ ($a = -7, b = -2$ where $x = a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 3 \end{bmatrix}$).

On the other hand

$\left[\begin{array}{cc|c} 0 & 2 & 1 \\ 1 & 3 & 0 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & -3/2 \\ 0 & 1 & 1/2 \end{array}\right]$ so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_C = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 0 & 2 & 1 \\ 1 & 3 & 1 \end{array}\right] \to \dots \to \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{array}\right]$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_C = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$

So $P_{C\leftarrow B} = \begin{bmatrix} -3/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ so

$[x]_C = P_{C\leftarrow B}[x]_B = \begin{bmatrix} -3/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$ indeed

2) • If (v_1, v_2) is an orthonormal basis of W ~~and (v_3) of $W^\perp$~~,

$$\mathrm{proj}_W(v) = (v \cdot v_1)\, v_1 + (v \cdot v_2)\, v_2$$

So
$$\begin{aligned}
\mathrm{proj}_W(u+v) &= ((u+v)\cdot v_1)\, v_1 + ((u+v)\cdot v_2)\, v_2 \\
&= (u\cdot v_1 + v \cdot v_1)\, v_1 + (u \cdot v_2 + v \cdot v_2)\, v_2 \\
&\cancel{= \mathrm{proj}_W(u)} \\
&= (u\cdot v_1)\, v_1 + (u \cdot v_2)\, v_2 + (v \cdot v_1)\, v_1 + (v\cdot v_2)\, v_2 \\
&= \mathrm{proj}_W(u) + \mathrm{proj}_W(v)
\end{aligned}$$

Similarly $\mathrm{proj}_W(\lambda u) = \lambda\, \mathrm{proj}_W(u)$ for all $u, v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ by multilinearity of dot product.

• $p(v_1) = v_1$, $p(v_2) = v_2$, $p(v_3) = 0$ so

$$[p]_C = [p]_{(v_1, v_2, v_3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

• $[p]_B = P_{B \leftarrow C}\, [p]_C\, P_{C \leftarrow B}$ ~~where~~.

Let us find some v_1, v_2, v_3 as announced.

$$\boxed{v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in W}$$

$v_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \in W$ but I want to replace it

by
$$v_2 - \mathrm{proj}_{v_1}(v_2) = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \in W \text{ and is } \perp \text{ to } v_1.$$

$$\boxed{v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

Finally, pick $v_3 = v_1 \wedge v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$$\boxed{v_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}$$

I'll take an <u>orthonormal</u> basis for convenience.

$$C = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix} \right).$$

So $P_{B\leftarrow C} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$ and is an <u>orthogonal</u> matrix.

We can now compute

$$[p]_{B\leftarrow B} = P_{B\leftarrow C}\,[p]_{C\leftarrow C}\,P_{C\leftarrow B}$$

$$= P_{B\leftarrow C} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&0 \end{bmatrix} P_{B\leftarrow C}^{T} \qquad \text{(because)}.$$

$$= \begin{bmatrix} 5/6 & 1/6 & -1/3 \\ 1/6 & 5/6 & 1/3 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}$$

Observe that $[p]_C$ is more convenient than $[p]_B$. Of course, because C contains meaningful vectors wrt p.

3) $A = \begin{bmatrix} 1&2&2\\2&1&2\\2&2&1 \end{bmatrix}$. One can find $\lambda_1 = 5$, $\lambda_2 = -1$ as eigenvalues.

$E_5 = \text{span}\left(\underset{v_1}{\begin{bmatrix}1\\1\\1\end{bmatrix}} \right)$ (as usual)

$E_{-1} = \text{span}\left(\underset{v_2}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}-1\\1\\0\end{bmatrix}} \right)$. By symmetry of A, $v_2 \perp v_1$ and $v_3 \perp v_1$.

We just need to replace v_3 by $\text{perp}_{v_2}(v_3) = \begin{bmatrix}-1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$$= \begin{bmatrix}-1/2\\1\\-1/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\2\\-1\end{bmatrix}$$

and normalize:

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \text{ and } D = \begin{bmatrix} 5&0&0\\0&-1&0\\0&0&-1 \end{bmatrix} \text{ <u>work</u>.}$$

(check it).